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Review of Lyapunov Methods for stability of Dynamic Systems and their Region of Attraction

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Abstract: Lyapunov Methods are widely used for proving stability of Non Linear System without solving the system. There are two main methods i.e.Lyapunov Direct method & Lyapunov Indirect Methods. This paper outlines the examples where Lyapunov methods are effective as well as the examples where the methods fails to conclude stability .

Key Words: : Equilibrium Point, Dynamic System , Lyapunov Equation ,Lyapunov Methods, Region of Attraction, Stability

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1 Introduction

Introduction : Lyapunov Functions are widely discussed and used in proving the stability of Dynamic System arising in the engineering and industrial problems. As it is tedious and Time consuming to find solution of Dynamic System , the Russian Mathematician Alexander Lyapunov introduced Lyapunov functions to prove stability. Its Versatility and simplicity has its application in almost every branch of Engineering. Its main application is to detect the domain of attraction which plays important role in control systems. It is revealed from so many examples[2], [17] that Linear System $\dot{X} = AX$, $X \in \mathbb{R}^n$ is stable if matrix A is Hurvitz Matrix , moreover its only equilibrium point is origin which is asymptotically stable (explained in section2) So proving stability for Non Autonomous Linear system is much easier than non linear systems. In fact Lyapunov function's role in this case is just to find Region of Attraction(R.A.) or Domain of attraction.

There are two main Lyapunov Methods (section 1) to prove stability. Lyapunov functions play very important role in case of Dynamic System particularly control systems Further stabilization of non linear system is estimated by linearizing it. [9], [10], [11] Also it is claimed one can not presume stabilization of finite dimensional non linear system for its infinite dimensional system.[8]. whereas various researches have been done in finding proper Lyapunov Function for a particular system,which is difficult task. In case of Control System as in [14] an adaptive tracking control Lyapunov function(CLF) is used which is quadratic in parameter error.Here the problem of adaptive stabilization is reduced by solving it recursively via backstepping . In [15] the CLF is computed systematically by Lyapunov Indirect method using Algebraic Riccati equation which can be generalized for higher order terms. However computational complexity appearing can be solved using computer aided software. In [16], Linear Algorithm with Numerical approach is used to construct Lyapunov Function which includes Sub-Markov matrix.

The following give Examples of linear and non linear systems

Linear Autonomous (Time invariant) System

$$\dot{X} = AX, \quad X \in \mathbb{R}^n \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Where A is a matrix, $A \in \mathbb{R}^{m \times m}$ may represent closed loop or open loop system

Linear Time Variant System

$$\dot{X} = A(t)X, \quad X \in \mathbb{R}^n \qquad \ldots\ldots\ldots\ldots(2)$$

Non linear time Variant System form

$$\dot{X} = F(t,X), \quad X \in \mathbb{R}^n \qquad \ldots\ldots\ldots\ldots(3)$$

Non Linear Control System

$$\dot{X} = f(x(t), u(t)) \qquad \ldots\ldots\ldots\ldots (4)$$

where x(t) is state variable and u(t) Is control .

Section 1 :

In this, few terms are defined and explained

1.1) Equilibrium Point : - Point $X_e \in \mathbb{R}^n$ is said to be Equilibrium point for any of 1) , 2), 3), 4) if $\dot{X}_e = 0$

(It is found by equating r.h.s of systems 1) ,2), 3) 4)to zero)

1.2)Stability : Stability of Equilibrium Point : An equilibrium point of system is said to be

1.2.1) **stable** if for $\varepsilon > 0$, there is $\delta > 0$ such that $\| X_e \| < \delta$ implies $\| X(t) \| < \varepsilon$ for all $t \geq t_0$

1.2.2) **Asymptotically stable** if it is stable and $X(t) \to 0$ as $t \to \infty$ where x(t) being solution of the system

1.2.3) **Globally Asymptotically stable(GAS) :** if it is asymptotically stable on $\mathbb{R}^n$

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1.2.4) **Exponentially stable** if there exists constants K_1, K_2 and λ Such that $\| X(t) \| < K_1 \| X(t_0) \| < e^{-\lambda(t-t0)}$ for all $\| X(t_0) \| < K_2$, for $K_1, K_2, \lambda > 0$

1.2.5) **Unstable** if it is not stable i.e. there exists $\varepsilon > 0$ such that for every $\xi > 0$ there is $X(t_0)$ with $\| X(t_0) \| < \xi$ and $\| X(t_1) \| > \varepsilon$ for some $t_1 > t_0$

1.2.6) **Completely unstable** if there exists $\varepsilon > 0$ such that for every $\xi > 0$ and for every $X(t_0)$ with $\| X(t_0) \| < \xi$ $\| X(t_1) \| \geq \varepsilon$ for some $t_1 > t_0$

The solution x(t) is called trajectory or motion flow.

1.3) Stability of equilibrium point with respect to the trajectory

1.3.1) **stability with respect to trajectory**

The equilibrium point X_e is stable if for given outer circle C with radius $\varepsilon>0$,there is inner circle C_1 with radius ξ_1 such that trajectory starting inside circle C_1 never leave outer circle C

1.3.2) **Asymptotic stability with respect to trajectory**

The equilibrium point X_e is Asymptotically stable if there is some circle C_2 with radius ξ_2 having same property as C_1 but in addition trajectory starting inside C_2 tends to X_e as $t\rightarrow\infty$.

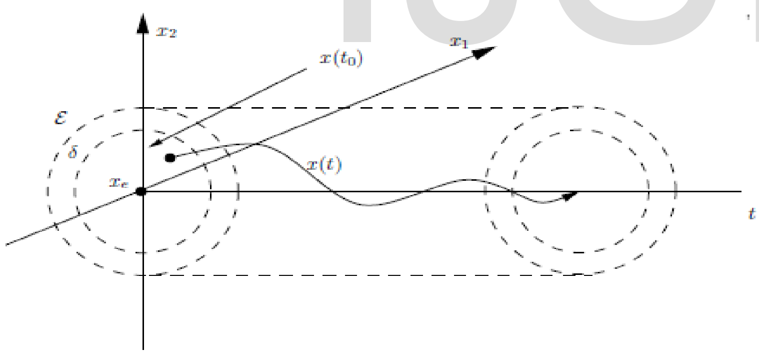

Figure 1. Asymptotic Stability of Trajectory

1.4) Boundedness

An equilibrium state X_e said to be bounded if there exists real constant M, which may depend on t_0 and $x(t_0)$ such that $\| X(t) \| \leq M$ for all $t \geq t_0$

1.5) What are the Lyapunov Functions

- The system can have more than one equilibrium points
- . Stability is the property of the equilibrium point and not of the system.
- Stability of of the equilibrium point is equivalent to the stability of the system if it has only one equilibrium point.
- The origin is only equilibrium point of system1)
- R.A. means Region of Attraction

1.5.1) Lyapunov Function Let $D \subseteq \mathbb{R}^n$ and let a scalar function $V: D\rightarrow \mathbb{R}$ be C^1 function & V be defined as

a) $V(x) \geq 0$ for all $x \in D-\{0\}$ and $V(x)=0$ if $x= 0$

b) $\dot{V}(x) \leq 0$ for all $x \in D$

c) $\dot{V}(x) < 0$ for all $x \in D-\{0\}$

Where V satisfying a), b) is called Candidate Lyapunov Function Any Equilibrium point of the system satisfying the conditions a), b) is **Lyapunov stable** and satisfying all a), b), c) is Asymptotically stable

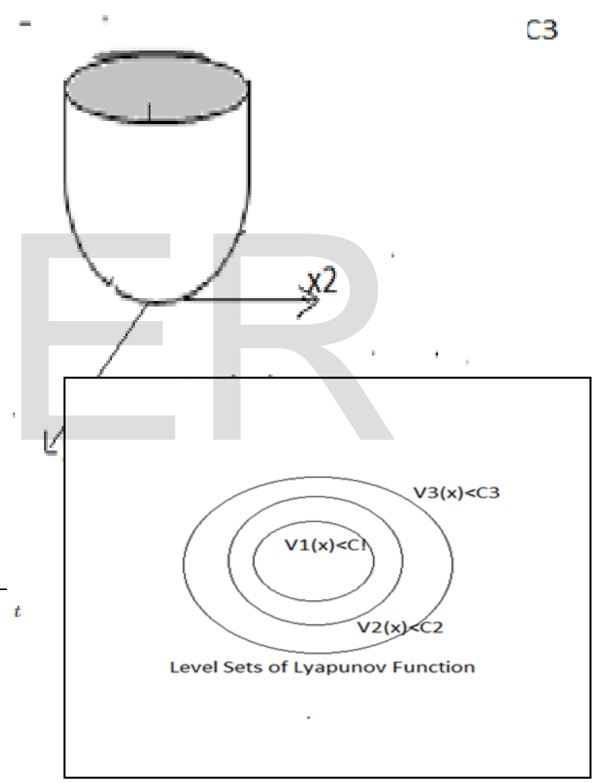

Figure 3.

Many times Energy Storage of the system considered to be Lyapunov function of the system.

Definition : A continuously Differentiable positive definite function V(x) is **Control Lyapunov Function** (C.L.F.) for the system $\dot{x} = f(x) + g(x)\,u$ if

- $\frac{\partial v}{\partial x} g(x) = 0$ for $x \in D$ for some $D \subseteq \mathbb{R}^n$ &
- $\frac{\partial v}{\partial x} .f(x)< 0$ for all $x\neq 0$

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∂x

- If $D = R^n$ then V is global Control Lyapunov Function

Lyapunov Function of dynamic Control System are Taken in terms of **Control Lyapunov function** (C.L.F)

There are two main methods to find Lyapunov Functions for given linear or Non linear system

i) **Direct Method** : In this, Function V(x) appears in Quadratic Form $V(x) = Ax_1^2 + B x_2^2$ where A,B> 0 ϵ |R constants and (x_1, x_2) ϵ $|R^2$ (same form can be generalized for $|R^n$)

Most often Non linear systems are linearalised By Jacobian Method and V(x) is found for its Linear form and then result is generalised for non linear system.

(some cases Quadratic form of Lyapunov Function is applied for non linear System)

ii) **Indirect method** :- In this the positive definite and symmetric Matrix P satisfying Lyapunov Equation

$\mathbf{A^TP + PA = - Q}$ is solved where P,Q, A have same matrix Dimensions and Most often Q is taken as Q= $\mathbf{I_n}$

(or Q is Positive definite ,symmetric matrix)

Once the Lyapunov equation is solved for P , V(x) assumes the quadratic form $V(x) = x^TP x$.

1.6) Region of Attraction(RA) :- Let $x^\cdot = f(t,x)$ .... be non linear time dependent system , where $f : D \rightarrow |R^n$, $D \underline{C} |R^n$

Let ϕ(t,x) be the solution of above system , then the set R= {x(t) / ϕ(t,x)→0 as t→∞} is called **Region of Attraction**

1.7) Invariant set : M is called Invariant set if for system

$x^\cdot = f(t,x)$ $x(0)$ ϵ M implies $x(t)$ ϵ M for all t >0

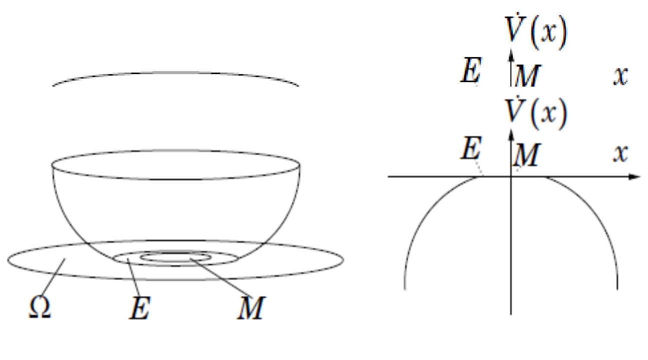

Figure 4. Invariant Set

1.8) La Salle's Invariant Set Theorem 1 :

- * Matrix $\mathbf{A}^{nxn}$ Hurvitz Matrix if its all eigen values have negative real Parts
  - * Linear System $X^\cdot = A X$, is G.A.S. if matrix **A** is Hurvitz
- * Here A = [-1 4; 0 -3] is matrix of order 2x2 with first row -1, 4 and 2nd row 0,-3
- * Main purpose of Lyapunov functions for linear system is to find region of attraction

Let Ω_l $\underline{C}$ $|R^n$ compact Invariant Set for $X^\cdot = f(t,X)$

Let V: $\Omega_l \rightarrow |R$ be C^1 function such that $V^\cdot(X) \leq 0$

for all X ϵ Ω_l, and E={ X ϵ Ω_l/ $V^\cdot(X) = 0$} and M be the largest subset of E ,then for all x(0) ϵ Ω_l, X(t) approaches to M as t→∞.

Section 2 : Different Examples including one linear and othe Non linear systems

linear System where Lyapunov function fits to Prove Global Asymptotic Stability (R.A. is $\mathbf{|R^2}$)

Example1)

Ex 1) Discuss the stability of the Linear System

$$\mathbf{\dot{x} = A\,x} \qquad \text{.............. .(E1)}$$

where A= [−1 4; 0 -3] and X=[x_1 x_2] ^{T} (x_1, x_2) ϵ $|R^2$

i.e. $x_1^\cdot = -x_1 + 4x_2$ & $x_2^\cdot = -3x_2$

By setting $x^\cdot = 0$ we find origin (0,0) is Equilibrium Point

Clearly the eigen values of A are {-1,-3} A is Hurvitz

To check the stability of Equilibrium Point .

<u>By direct Method</u> : Assume $V(x) = Ax_1^2 + Bx_2^2$

where A,B>0

$V^\cdot(x) = 2Ax_1x_1^\cdot + 2B\,x_2\,x_2^\cdot = 2Ax_1(-x_1 + 4x_2) + 2B\,x_2(-3x_2)$

$= -(2\,Ax_1^2 + 6Bx_2^2) + 8A\,x_1x_2 < 0$ for all $(x_1, x_2) \neq (0,0)$ & for all real A,B>0

Therefore origin is Asymptotically stable and the Region of asymptotic Stability is being entire $|R^2$ set, origin is G.A.S.

<u>By Indirect Lyapunov Method</u>: -Here, to find Matrix P of same dimesion as that of A and which is Positive definite also , satisfying

$A^TP + PA = -Q$ where Q= I_2. And then find $V(x) = X^TPX$

Solving for P , P= [1/2 ½; 1/2 5/6]

So $V(X) = X^TPX$ = [x_1 x_2] [1/2 1/2 ; 1/2 5/6] [x_1 x_2] ^{T}

$= 1/2\; x_1^2 + x_1x_2 + 5/6\, x_2^2$

$V^\cdot(X) = 1/2\;(2x_1x_1^\cdot) + x_1^\cdot x_2 + x_1\,x_2^\cdot + 5/6\,(2x_2\,x_2^\cdot)$

$= -x_1^2 + 4x_1x_2 - x_1x_2 + 4x_2^2 - 3x_1x_2 - 5x_2^2 = -(x_1^2 + x_2^2)$

< 0 for all $(x_1, x_2) \neq (0,0)$

Clearly it shows the domain of asymptotic Stability is entire $|R^2$ set (also it is clear from eigen values of matrix A That (0,0) is asymptotically stable)

In this case either method can be used as shows similar result about stability.

The Non linear System where Lyapunov Function can not conclude Asymtotic Stability unless damping torgue is added [3]

Example 2) The dynamic equation of a pendulum comprising a mass M at the end of a rigid but massless rod of length R is $MR\ddot{\theta} + Mg \sin\theta = 0$

where θ is the angle made with the downward direction, and g is the acceleration due to gravity. Consider the system in state-space form, (without damping)

$MR\ddot{\theta} + Mg \sin\theta = 0$ ......(5)

let $x_1 = \theta$ and $x_2 = \dot{\theta}$ then $\dot{x}_1 = x_2$ & $\dot{x}_2 = -g/R \sin x_1$

Taking candidate Lyapunov function as total Energy in the system. Then $V = 1/2(MR^2 x_2^2) + MgR(1-\cos x_1)$ =Kinetic + Potential Eenergy ,

$$\dot{V} = MR^2 x_2 (-g/R \sin x_1) + MgR \sin x_1 . x_2 = 0$$

Hence, V is a Lyapunov function and the system is stable in sense of Lyapunov .We cannot conclude asymptotic stability with this analysis.

Now let the damping torque **$-Dx_2$** be added to the velocity $\dot{x}_2$ so that the new equations are

$\dot{x}_1 = x_2$ and $\dot{x}_2 = -Dx_2 - g/R \sin x_1$ with same $V(x)$ so that $\dot{V} = -DMR^2 x_2^2 < 0$ for $x_2 \neq 0$

From this we can conclude stability Lyapunov Stable, but cannot directly conclude asymptotic stability.

Notice however that $\dot{V} = 0. \rightarrow x_2 = 0 \rightarrow \dot{\theta} = 0$ Under this condition, $\ddot{\theta} = -g/R \sin\theta$. Hence $\ddot{\theta} \neq 0$ for some

$\theta \neq k\pi$ for integer k, i.e. if the pendulum is not vertically down or vertically up. This implies that, unless we are at the bottom or top with zero velocity, we shall have $\ddot{\theta} \neq 0$ when $\dot{V} = 0$., so $\dot{\theta}$ will not remain at 0, and hence the Lyapunov function will begin to decrease again. The only place the system can end up, therefore, is with zero velocity, hanging vertically down or standing vertically up, i.e. at one of the two equilibria. The formal proof of this result in the general case (LaSalle's invariant set theorem)

The conclusion of local asymptotic stability can also be obtained directly through an alternative choice of Lyapunov function. Consider the Lyapunov function candidate

$V(x) = 1/2\, x_2^2 + 1/2\,(x_1 + x_2)^2 + 2(1 - \cos x_1)$

$\dot{V} = -(x_2^2 + x_1 \sin x_1) = -(\dot{\theta}^2 + \theta \sin\theta) \leq 0$

Also, $\dot{\theta}^2 + \theta \sin\theta = 0 \rightarrow \dot{\theta}^2 = 0$,$\theta \sin\theta = 0 \rightarrow \dot{\theta} = 0$;

$\theta = 0$; Hence, $\dot{V}$ is strictly negative in a small neighborhood around 0. This proves asymptotic stability.In above, **Energy storage function is taken as Lyapunov function which does not conclude asymptotic stability Where as when quadratic form of Lyapunov function is used it shows asymptotic stability**.

Theorem 2 *Consider the nonlinear time-invariant system defined on a Banach space X with norm $\|\ \|$*

Consider $\dot{z}(t) = F(z(t))$, ...(*) $t > 0$ with $z(0) = z_0$ where z_0 is the initial condition, the nonlinear operator $F : D(F) \subset X \rightarrow X$ is densely defined on X. Assume that this system is well-posed; that is, it has a unique solution that can be written $z(t) = S(t) z_0$, where $S(t)$ is a nonlinear C_0-semigroup on X generated by the operator F. finite-dimensional. Assume also that F is differentiable and define

$A = \left.\dfrac{\partial F}{\partial z}\right|_{z=z_e}$ *to be the linearization of (*). Then*

(i) if $Re\,\lambda(A) < 0$ for all $\lambda \in \sigma(A)$, then the equilibrium z_e to (*) is exponentially stable where $\sigma(A)$ is the spectrum of A.

(2) if there exists $\lambda \in \sigma(A)$ such that $Re\,\lambda > 0$, then the equilibrium z_e to (*) is unstable.

Linearization of Non linear System using Jacobian to check stability by Lyapunov Methods [2]

Example 3) Consider Non linear System(Van Der Pol equation)

$$\dot{x}_1 = x_2 \ \& \ \dot{x}_2 = -x_1 + \varepsilon x_2 - \varepsilon x_1^2 \quad \text{.........E3)}$$

clearly (0,0) is equilibrium point of the system ,taking Jacobian around (0,0) , the linearised system is

$\dot{x} = [\dot{x}_1\ \dot{x}_2]^T = Ax$ where A= [0 1; -1 ε]

Let the Lyapunov function be $V = x_1^2 + a\, x_2^2$

$$\dot{V} = 2 x_1 \dot{x}_1 + 2a\, x_2 \dot{x}_2 = 2 x_1 (x_2) + 2 a x_2(-x_1 + \varepsilon x_2)$$

So, $\dot{V} = 2x_1x_2(1-a) + 2a\,\varepsilon\, x_2^2 \leq 0$

Setting a= 1 and $\varepsilon < 0$

As linearized system depends on' a' & 'ε' ,

So system (0,0) is asymptotically stable if a= 1 & $\varepsilon < 0$

The state System strongly depends on nature of ε i.e. system equilibrium point is asymptotically stable If $\varepsilon < 0$, ii) If $\varepsilon > 0$ unstable iii) Lyapunov stable if $\varepsilon = 0$

But original system (E5)can not guarantee stability of (0,0) even $\varepsilon < 0$ as the Jacobian of the system $\frac{\partial f}{\partial x}$ = [0 1; 1 ε] can have some of the eigen values $(\varepsilon \pm \sqrt{\varepsilon^2+4})/2$ in right plane i.e. (Re(λA) >0 for some $\varepsilon < 0$ So clearly stability of linear system can not be concluded for non linear system

Example of Dynamic System where Lyapunov Methods give different Region of Attraction [9]

Example 4) Consider following non linear system

$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \approx f_1(x_1, x_2)$ &

$\dot{x}_2 = 4x_1^2 x_2 + x_2(x_1^2 + x_2^2 - 2) \approx f_2(x_1, x_2)$ ....E4)

clearly $f_1(0,0) = f_2(0,0) = 0$ i.e. x=0 is equilibrium p oint

Candidate Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$

Which is globally positive definite has

$\dot{V}(x_1, x_2) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0$ if $x_1^2 + x_2^2 < 2$

So Domain of attraction (R.A.) can be taken as Ball B_ε

Such that $B_\varepsilon = \{ \|x\| < \varepsilon / 0 < \varepsilon < 2\}$

Now By Lyapunov Indirect Method , linearizing the system

By Jacobian $\frac{\partial F}{\partial x(0,0)} = [-2\ 0; 0\ -2] = A$ and finding Matrix P >0

Such that $A^T P + PA = -Q$ where Q= [1 0 ; 0 1]

P= [1/4 0 ; 0 1/4] which is positive definite.

Taking $V = [x_1; x_2]^T P [x_1; x_2] = 1/4 x_1^2 + 1/4 x_2^2$

$\rightarrow \dot{V} = -(x_1^2 + x_2^2) < 0$ for all $(x_1, x_2) \neq (0,0)$

Proves the global asymptotic stability of (0,0) which gives Domain of attraction as entire $\mathbb{R}^2$

***Theorem 3**: Consider the nonlinear system (*) defined on a Banach space X. Assume that the nonlinear operator $F : D(F) \subset X \rightarrow X$ generates a nonlinear C_0-semigroup S (t).*

Let z_e be an equilibrium for the above system (*) and suppose that S (t) is Fr´echet differentiable at z_e.

(i) If z_e is an exponentially stable equilibrium of the lnearized system, then z_e is a locally exponentiallystable equilibrium of the nonlinear system (*).

(ii) If the linearized system is unstable, then the nonlinear system (*) is locally unstable.

Example where asymptotic stability of linear system does not conclude A.S. to the infinite dimensional non linear system [8]

Eample 5) Let $\ell 2$ be the space of square summable sequences and N the set of natural numbers with norm $\| \cdot \|_{\ell 2}$.

For any $z(t) = (z_1(t), z_2(t), \ldots, z_n(t), \ldots) \in \ell 2$ with $n \in N$,

consider $\dot{z}_n = -1/z_n + z_n^2$ ………….......(E5)

Then this system has infinitely many equilibrium points ,

The set of euilibria is $E = \{ z \in \ell 2 |\ z_n \in \{0, 1/n\}, n \in N \}$

Jacobian is defined as $\frac{\partial F(0,0)}{\partial x} = [\frac{\partial F}{\partial x_1}\ \ \frac{\partial F}{\partial x_2}]$,$(x_1, x_2) \in \mathbb{R}^2$

If the eigen values of Jacobian Matrix of the system are

- Negative or complex with negative real part the Equlibrium point is called **Sink** .The system is stable
- positive or complex with positive real part the Equlibrium point is called **Source .**
- The system is unstable with real of different sign, equilibrium point is called **Saddle point** . The system is unstable

Linearizing the system around $z_e = 0$,

$\dot{z}_n(t) = -1/n\ z_n(t)$, $t \geq 0$ which has solution

which has solution $z(t) = (z_1(0)e^{-t}, z_2(0)e^{-1/2t}, \ldots)$.

The linearized system (E5) is asymptotically stable since

$\lim_{t\to\infty} \|z(t) - z_e\|_{\ell 2} = \lim_{t\to\infty} \|z(t)\|_{\ell 2} =$

$\lim_{t\to\infty} \left(\sum_{n=1}^{\infty} Zn2^* \ e^{(-2/n)t} \right) = 0$

The solution of (E5) is $z_n(t) = \frac{z_{0^*} * e^{(-1/n)t}}{z_{0n}(-1 + e^{(-1/n)t}) + 1}$.

Where z_{0n} is initial condtion. For any $\partial > 0$, choosing N such that $1/n < \partial$

In the nonlinear system (E5), choose components of the initial condition z_0 to be zero except in the nth position, which is chosen to be 1/n . that is $z_0 = (0,0,0,\ldots 1/n, 0, 0 \ldots)$

Given this initial condition, the solution to (4) is

.z(t) = (0,0,0,....1/n, 0 ,0 ....)

and hence $\|z_0 - z_e\|_{\ell 2} = 1/n < \delta$. However,

$\lim_{t\to\infty} \|z(t) - z_e\|_{k\ell 2} = 1/n \neq 0$

hence the zero equilibrium of non linear system is not asymptotically stable .

Lyapunov-Like Lemma :

- *If a scalar function V = V(t,x) is such that V(t,x) is lower bounded*
- *$\dot{V}(t,x)$ is negative semi definite along the trajectories of $\dot{x} = f(t,x)$ and*
- *$\dot{V}(t,x)$ is uniformly continuous in time then*

$\dot{V}(t,x) \rightarrow 0$ as $t \rightarrow \infty$

Section3

Control Lyapunov function for non linear control System [10]

Example 6) Consider Non Linear system

$\dot{x} = f(x) + g(x) \cdot u$, ……….(E6) $f(0) = 0$

where f(x) is input and 'u' is control Let $f(x) = -x^3$ and let $g(x) = 1$ Therefore $\dot{x} = -x^3 + 1.\ u$

Taking L.F. $V(x) = ½\ x^2$ then V is positive definite and

$\dot{V} = x.\dot{x} = -x^4 + u.x$

To get $\dot{V}$ Negative definite ,

choosing $u = x^3 - k.x$ where 'k' is positive constant

$\dot{V} = -k\,x^2 < 0$ for all $x \neq 0$

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Which proves system is G.A.S.

However control 'u' can be taken of any other suitable choice .

Example of non autonomous sytem where LaSalle 's Invariant set Theorem does not work

Example 7) For closed loop error Dynamics of an adaptive Control system $\dot{e} = -e + \theta w(t)$

$$\& \quad \dot{\theta} = -e\, w(t) \qquad \ldots\ldots\} \quad (E7)$$

Where 'e' represents the tracking error and w(t) is a bounded function of time t. .This function w(t) makes the system non autonomous.

Taking Lyapunov Function for above system ,

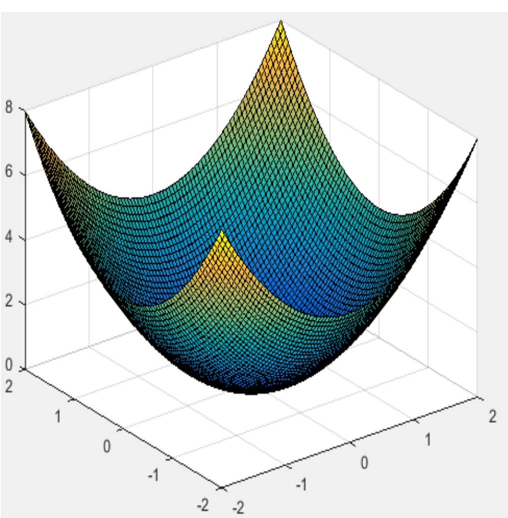
$V(e, \theta) = e^2 + \theta^2$, so that its time derivative about system Trajectory is $\dot{V}(e, \theta) = 2 e \dot{e} + 2 \theta \dot{\theta}$

$$= 2 e(-e + \theta w(t)) + 2 \theta (- e w(t)) = -2 e^2 \leq 0$$

This implies V is decreasing function of time and both e(t) & θ (t) are bounded . But due to non autonomous nature of system dynamics, LaSalle theorem can not be used to concude the convergence of e(t) to the origin

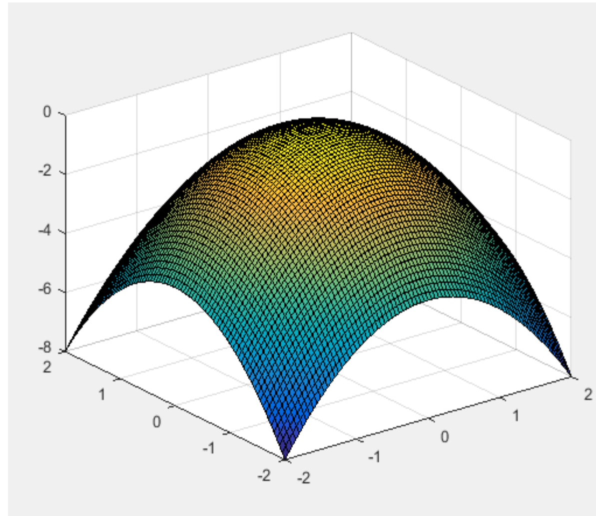
Now $\ddot{V} = -4 e \dot{e} = -4 e((-e + \theta w(t))$

Since w(t) is bounded by hypothesis and also e(t) & θ (t) are bounded , it is clear that $\ddot{V}$ is bounded . Hence $\dot{V}$ Is uniformly continuous and by Lyapunov-Like Lemma, $\dot{V} \rightarrow 0$ Which indicates tracking error $e(t) \rightarrow 0$ as $t \rightarrow \infty$



A Typical Lyapunov Function in Quadratic Form

v(x)



Lyapunov Derivative $\dot{v}$(x)

Conclusion :

Proving stability of linear system is easier by any Lyapunov method , in fact the stability is clear from system matrix A if it is Hurvitz . For non linear systems , their linearization is checked for stability and the result is generalised for non linear system (i.e. if linear system is stable then so its corresponding non linear.) But some cases such generalization backfires.

Normally Lyapunov Direct Method proves to be a more general and powerful approach, enabling the potential global stability of the general nonlinear system to be investigated and therefore does not suffer from the drawbacks incurred by Lyapunov's indirect method

Provided that the required Lyapunov function, as well as its time derivative, must satisfy rigid constraints.

Furthermore, there can be 'n' Lyapunov functions suitable to the system and failing of one Lyapunov function to prove stability of the system does not concludes that system is unstable. It only suggests pertaining to particular L.F. system can be unstable.

So Lyapunov functions satisfy sufficient condition and not necessary condition to prove stability .

Main application of Lyapunov function is to find Domain of attraction which is of paramount use in Dynamic systems such as control system.

However there is need to formulate Lyapunov function in future so as to facilitate uniformity in its applications.

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